9 November 2007

Columbia University in the City of New York

Equity Correlation Swaps: A New Approach For Modelling & Pricing

14th Annual CAP Workshop on Derivative Securities and Risk Management
New York City

Sebastien Bossu
Equity Derivatives Structuring — London

Dresdner Kleinwort

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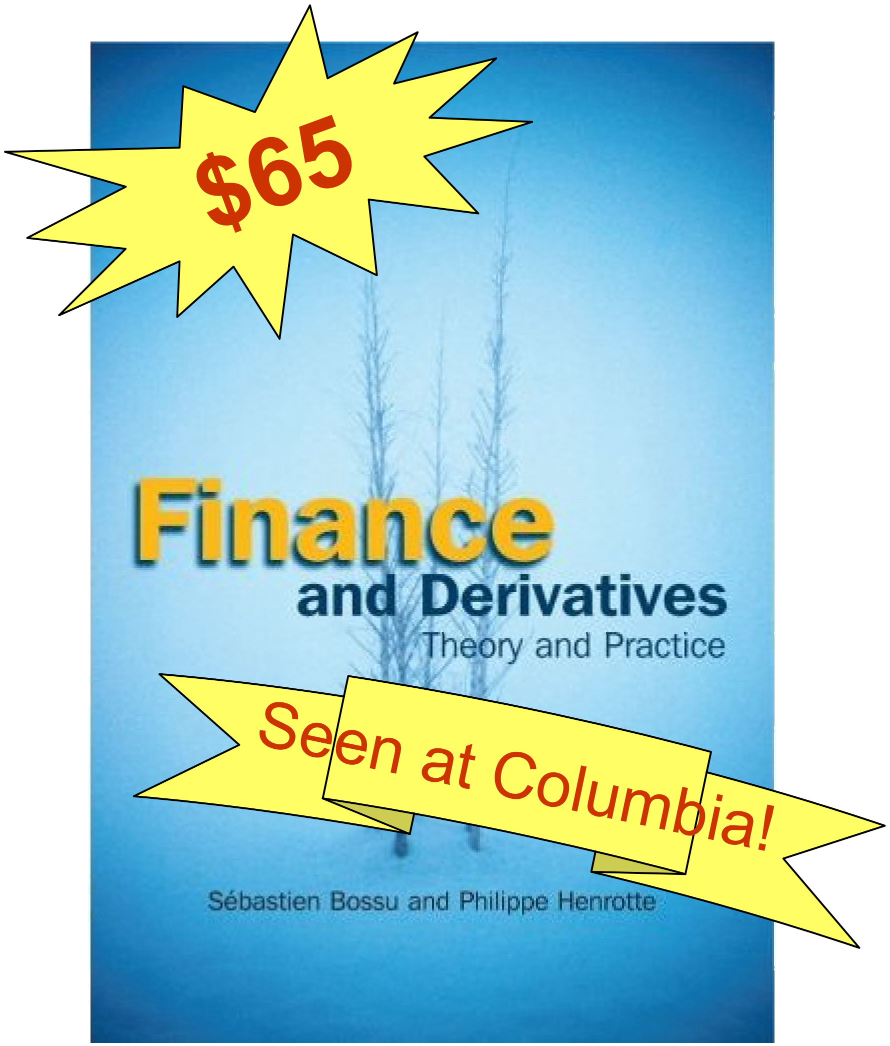
Blurb: *F&D* (Wiley) — S. Bossu, Ph. Henrotte



“ Finance and Derivatives teaches all of the fundamentals of quantitative finance clearly and concisely without going into unnecessary technicalities. You'll pick up the most important theoretical concepts, tools and vocabulary without getting bogged down in arcane derivations or enigmatic theoretical considerations. ”

– Paul Wilmott

Agenda

1. Fundamentals of index variance, constituent variance and correlation
2. Toy model for derivatives on realised variance
3. Rational pricing of correlation swaps

1. Fundamentals of index variance, constituent variance and correlation

1. Fundamentals of index variance, constituent variance and correlation

1.1 Realised and Implied Correlation

1.2 Correlation Proxy

1.3 Application: Variance Dispersion Trading

Realised and Implied Correlation

- **Realised Correlation**
  - **Pair of stocks**: statistical **coefficient of correlation** between the two time series of daily log-returns
  - **Basket of N stocks**: **average** of the N(N-1)/2 pair-wise correlation coefficients
- **Implied Correlation**
  - **Pair of stocks**: usually unobservable
  - **Basket of N stocks**: occasionally observable through quotes on basket calls or puts from exotic desks
  - **Liquid indices**: observable for listed strikes and maturities

Realised and Implied Correlation

- **<u>Realised Correlation Definitions</u>** <u>(Equal Weights Assumption)</u>
  - **Average pair-wise** ('naive') definition:

$$\rho_{\text{Pairwise}} \equiv \frac{2}{N(N-1)} \sum_{i<j} \rho_{i,j}$$

  - **Canonical** (econometric) definition:

$$\rho_{\text{Canonical}} \equiv \frac{\sigma^2_{\text{Index}} - \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2}{\boxed{\bar{\sigma}^2_{\text{Constituent}}} - \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2} \underset{<}{\approx} \frac{\sum_{i<j}\sigma_i\sigma_j\rho_{i,j}}{\sum_{i<j}\sigma_i\sigma_j}$$

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 \underset{>}{\approx} \left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2$$

Realised and Implied Correlation

- **Implied Correlation Definition** (Equal Weights Assumption)
  - **No 'naive' definition** (pair-wise implied correlations not observable)
  - **Canonical** (econometric) definition:

$$\rho^*_{\text{Canonical}} \equiv \frac{\sigma^{*2}_{\text{Index}} - \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^{*2}}{\overline{\sigma}^{*2}_{\text{Constituent}} - \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^{*2}}$$

    - Note that the implied volatility surface translates into an **implied correlation surface**. We use fair variance swap strikes for σ^*'s unless mentioned otherwise.

1. Fundamentals of index variance, constituent variance and correlation

Correlation Proxy

- The previous definitions are easily generalised to arbitrary index weights
- **<u>Proxy Formula:</u>** Under certain regularity conditions on the weights, residual volatility becomes negligible and we have:

$$\begin{cases} \rho_{\text{Canonical}} \xrightarrow[N \to +\infty]{} \left(\dfrac{\sigma_{\text{Index}}}{\overline{\sigma}_{\text{Constituent}}} \right)^2 \equiv \hat{\rho} \\ \rho^*_{\text{Canonical}} \xrightarrow[N \to +\infty]{} \left(\dfrac{\sigma^*_{\text{Index}}}{\overline{\sigma}^*_{\text{Constituent}}} \right)^2 \equiv \hat{\rho}^* \end{cases}$$

- Condition: $\dfrac{\text{MaxWeight}}{\text{MinWeight}} = o\left(\sqrt{N}\right)$

Correlation Proxy

- Correlation (realised and implied) is thus close to the **ratio** of **index variance** to **average constituent variance**

$$\text{Correlation} \approx \frac{\text{Index Variance}}{\text{Average Constituent Variance}}$$

- This is interesting because **index variance** and **average constituent variance can be traded on the OTC variance swap market**

1. Fundamentals of index variance, constituent variance and correlation

1.1 Realised and Implied Correlation

1.2 Correlation Proxy

1.3 Application: Variance Dispersion Trading

Application: Variance Dispersion Trading

- **<u>Variance Dispersion Trades</u>**
  Spread of variance swap positions between an index and its constituents, usually:
  - **Long Average Constituent Variance**
  - **Short Index Variance**

- $$\text{Payoff} = \overline{\sigma}^{2}_{\text{Constituent}} - \sigma^{2}_{\text{Index}} = \overline{\sigma}^{2}_{\text{Constituent}} \times \left[1 - \hat{\rho}\right] \geq 0$$

- $$\text{Cost} = \overline{\sigma}^{*2}_{\text{Constituent}} - \sigma^{*2}_{\text{Index}} = \overline{\sigma}^{*2}_{\text{Constituent}} \times \left[1 - \hat{\rho}^{*}\right] \geq 0$$

- Exposure: long volatility, short correlation

Application: Variance Dispersion Trading

- By **underweighting the constituents' leg** with a factor $\beta = \rho^* < 1$, several benefits are obtained:
  - **Vega-Neutrality**
    On trade date, if constituent variance goes up 1 point and implied correlation is unchanged, index variance would go up by ρ^* points and the P&L is: β **x 1pt –** ρ^* **pts = 0**
  - **Zero cost**

$$\text{Cost} = \beta \overline{\sigma}^{*2}_{\text{Constituent}} - \sigma^{*2}_{\text{Index}} = 0$$

  - **Straightforward p&l decomposition**

$$\text{P\&L} = \text{Payoff} - \underbrace{\text{Cost}}_{\text{Zero}} = \overline{\sigma}^{2}_{\text{Constituent}} \times [\underbrace{\hat{\rho}^*}_{\beta} - \hat{\rho}]$$

2. Toy Model for Derivatives on Realised Variance

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Realised Variance: A Tradable Asset

- **Variance Swap**
  At expiry two parties exchange the **realised variance** of e.g. DJ EuroStoxx 50 daily log-returns, against a strike ('implied variance')
- OTC market has become very liquid on S&P 500 and DJ EuroStoxx 50, with bid-offer spreads sometimes as tight as ¼ vega.
- CBOE introduced Three-Month Variance Futures on the S&P 500 in 2004.

2. Toy Model for Derivatives on Realised Variance

Which Model for Realised Variance?

Fischer Black:

'I start with the view that nothing is really constant. Volatilities themselves are not constant, and **we can't write down the process by which the volatilities change with any assurance that the process itself will stay fixed**. We'll have to keep updating our description of the process.'

'Studies of Stock Price Volatility Changes', cited in Fischer Black and the Revolutionary Idea of Finance, P. Mehrling, John Wiley & Sons, 2005

Toy Model for Realised Variance

- Popular models (in particular Heston) for volatility or variance focus on the **instantaneous, non-tradable volatility**
- Other approaches (Dupire, Buehler) focus on the **variance swap curve**, which is tradable; or a **fixed-term variance asset** (Duanmu, Carr-Sun)
- **<u>Toy Model</u>**
  Straightforward modification of Black-Scholes where the volatility of the variance asset v_t linearly collapses as we approach its expiry T:

Volatility of volatility

$$\frac{dv_t}{v_t} = 2\omega \frac{T-t}{T} dZ_t$$

Toy Model for Realised Variance

- $\mathbf{v_T}$ is the price of the variance asset at expiry and coincides with **realised variance** over the interval [0, T]
- $\mathbf{v_0}$ is the **fair price of the variance asset** which can be observed on the variance swap market or calculated through the replicating portfolio of puts and calls
- $\mathbf{v_0 = E(\, v_T \,)}$
- The terminal distribution of v_T is **lognormal**, making closed-form formulas for European derivatives on realised variance easy to derive

2. Toy Model for Derivatives on Realised Variance

Application: Volatility Swap

- Payoff = $\sqrt{v_T} - K_{vol}$
- With the Toy Model we find:

$$K_{vol} = \underbrace{\boxed{\sqrt{v_0}}}_{\text{Variance Swap Strike}} \underbrace{\boxed{\exp\left(-\frac{1}{6}\omega^2 T\right)}}_{\text{Quadratic Adjustment}}$$

- Numerical example: $v_0 = 20^2 = 400$, $T = 1$, $\omega = 50\%$
  $\rightarrow K_{vol} \approx 19.2$

2. Toy Model for Derivatives on Realised Variance

Parameter Estimation

- **Implied approach**
  Knowing K_{vol} and K_{var} (= v_0), we can back out an implied volatility of volatility parameter:

$$\hat{\omega}_{\text{Implied}} = \sqrt{\frac{6}{T} \ln \frac{K_{var}}{K_{vol}}}$$

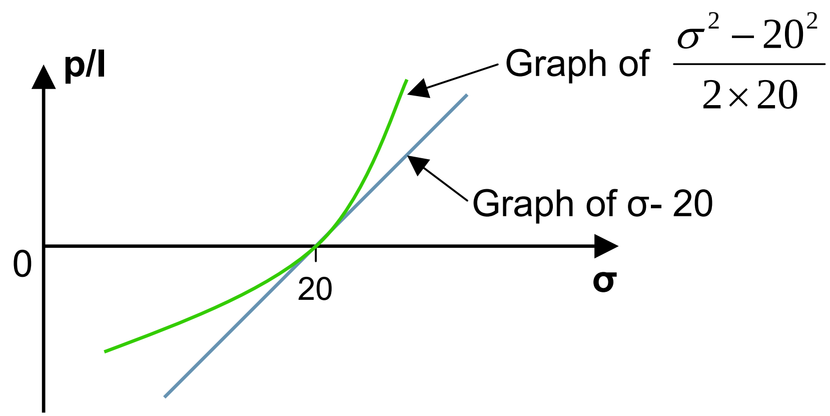
- Numerical example (DAX):
  $T = 1$
  $K_{var} \approx$ VDAX New = 17.75
  $K_{vol} \approx$ ATM Vol = 17
  - $\omega = [6 \times \ln(17.75/17)]^{1/2} = 50.9\%$

Parameter Estimation

- **Historical approaches**
  - **Classical**: e.g. reconstitute historical time series of fixed-maturity variance prices $(v_t)_{0 \le t \le T}$, on a rolling basis (computationally intensive)
  - **Break-even historical analysis**: e.g. find the quadratic adjustment which, on average, neutralises the P&L of an arbitrageur trading the spread between variance and volatility swaps

Parameter Estimation: Break-Even Analysis

- **If volatility and variance swaps had the same strike, there would be an arbitrage**:



- Thus $\mathbf{K_{vol} < K_{var}}$. Consider an arbitrageur who executes on dates m = 1,2,...,M a series of normalised spread trades: **BUY** $1/(2K_m^2)$ units of variance at K_m and **SELL** $(1/K_m)$ units of volatility at K_m/γ:

$$p/l = \sum_{m=1}^{M} \left[\left(\frac{R_m^2 - K_m^2}{2K_m^2} \right) - \left(\frac{R_m - (K_m/\gamma)}{K_m} \right) \right]$$

where R_m denotes realised volatility between dates m and m + τ

Parameter Estimation: Break-Even Analysis

- Assuming p/l = 0 and solving for γ, we find:

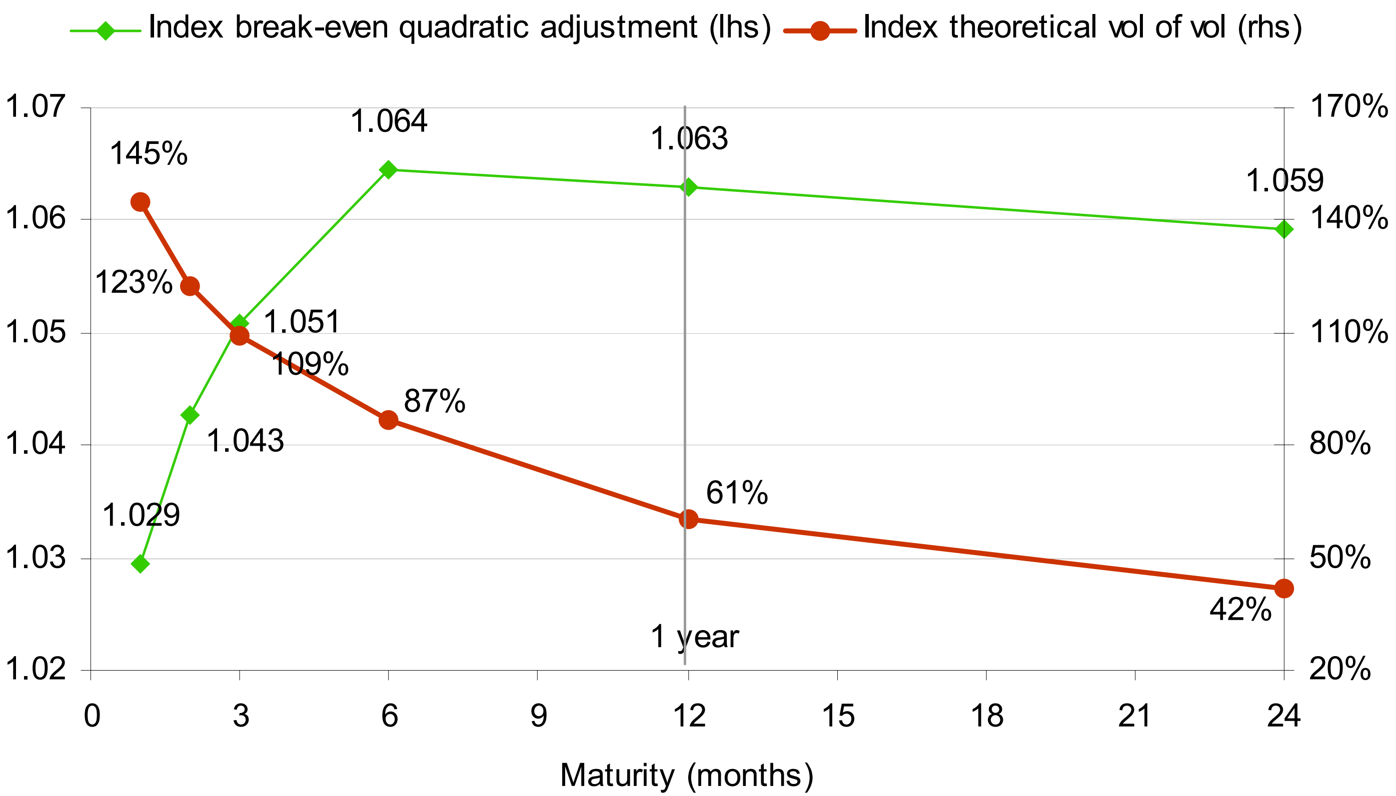
$$\gamma = \left[1 - \frac{1}{2M}\sum_{m=1}^{M}\left(\frac{R_m - K_m}{K_m}\right)^2\right]^{-1} \equiv \hat{\Gamma}$$

- This is the **break-even quadratic adjustment**. The corresponding theoretical volatility of volatility parameter is then given as:

$$\hat{\omega}_{\text{Implied}} = \sqrt{\frac{6}{T}\ln\hat{\Gamma}}$$

Parameter Estimation: Break-Even Analysis

- Results for the Dow Jones Euro Stoxx 50 index, using monthly trading dates m between 2000 and 2005



2. Toy Model for Derivatives on Realised Variance

Model Limitations

- The usual Black-Scholes limitations apply: constant volatility of volatility, no transaction costs, continuous hedging.
- **Specific limitations:**
  - **Log-normal assumption inconsistent with additivity of variance**: the toy model is not suitable to model the variance swap curve, even with a time-dependent ω
  - **No joint dynamics with the asset price process**: the toy model does not explain/take into account the equity skew
  - **Consistency with vanilla option prices not considered**.

3. Rational Pricing of Correlation Swaps

3. Rational Pricing of Correlation Swaps

Correlation Swaps

- **Correlation Swap**
  At maturity two parties exchange the **average pair-wise realised correlation** between e.g. the DJ EuroStoxx 50 constituents, against a strike.
- OTC market, not very liquid. Introduced in early 2000's as a means for equity exotic desks to recycle their correlation parametric risk.
- Typically **correlation swaps trade at a strike which is 5 to 15 points below implied correlation**.

Correlation Swaps

- Correlation Swap Payoff:

$$Payoff \equiv \frac{2}{N(N-1)} \sum_{i<j} \rho_{i,j} = \rho_{\text{Pairwise}}$$

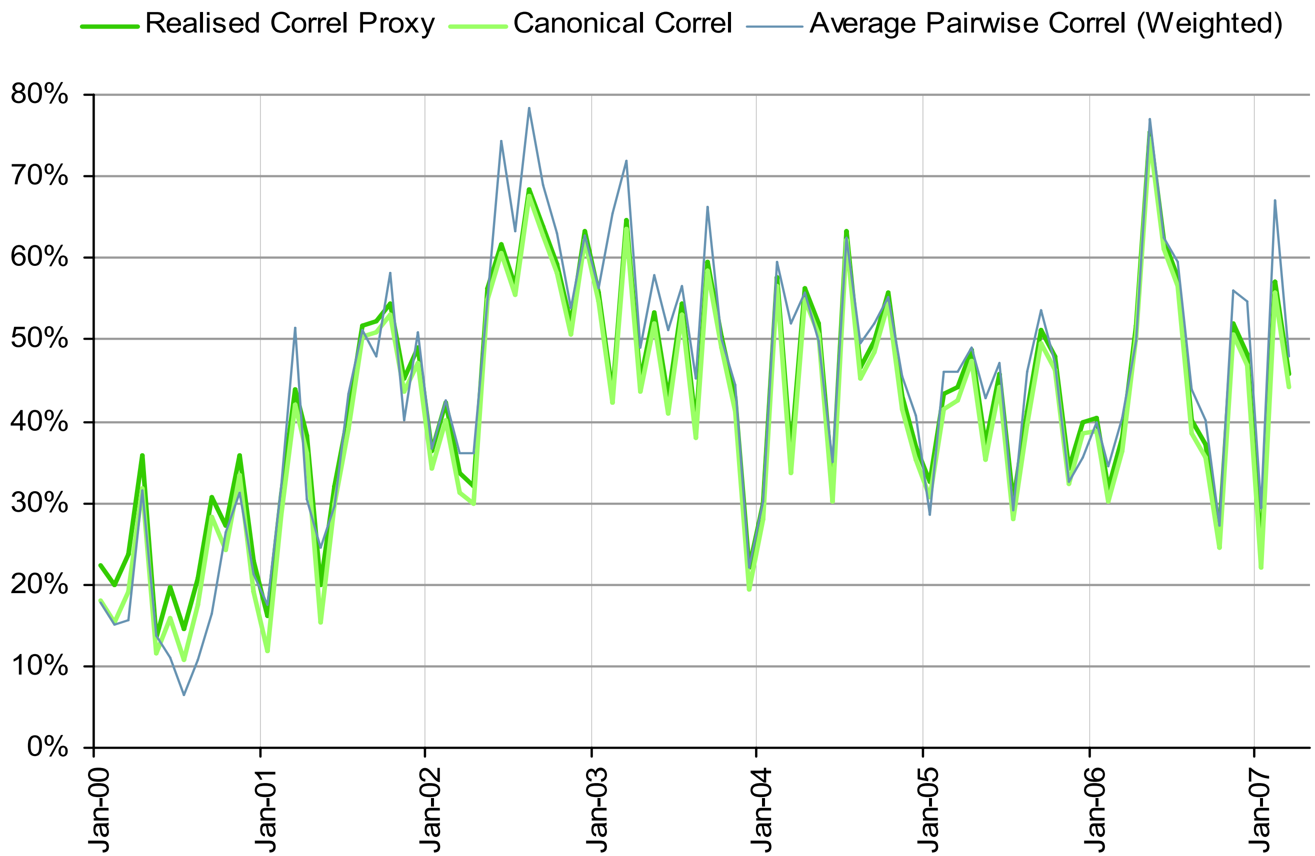
- The **pricing** and **dynamic hedging** of this payoff is non-trivial. However we can simplify the problem using the Proxy formulas:

$$Payoff \approx \rho_{\text{Canonical}} \approx \hat{\rho} = \frac{\sigma^2_{\text{Index}}}{\overline{\sigma}^2_{\text{Constituent}}}$$

which is the ratio of **two tradable assets**: index variance and average constituent variance

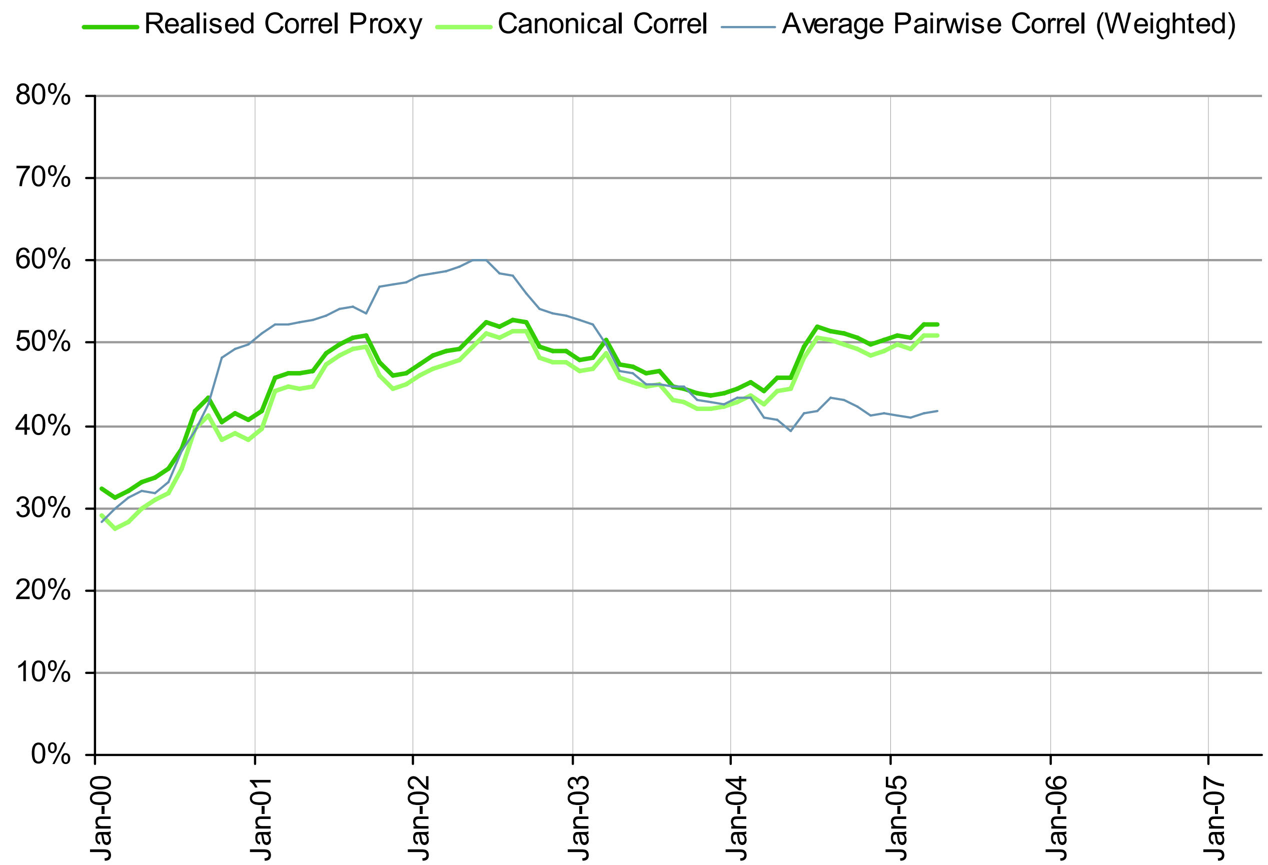
How Good Is The Proxy?

- 1-month realised correlation



How Good Is The Proxy?

- 24-month realised correlation



3. Rational Pricing of Correlation Swaps

Two-factor Toy Model

- Define $\mathbf{v_t^I}$ as the **index variance** asset, $\mathbf{v_t^S}$ as the **average constituent variance** asset, with the following forward-neutral dynamics:

Volatility of index volatility

$$\frac{dv_t^{\mathrm{I}}}{v_t^{\mathrm{I}}} = 2\boxed{\omega_{\mathrm{I}}}\frac{T-t}{T}dW_t$$

$$\frac{d\bar{v}_t^{\mathrm{S}}}{\bar{v}_t^{\mathrm{S}}} = 2\boxed{\overline{\omega}_{\mathrm{S}}}\frac{T-t}{T}\left[\boxed{\chi}dW_t + \sqrt{1-\boxed{\chi}^2}\,dZ_t\right]$$

Volatility of constituent volatility

Correlation between index and constituent vols

- Define $c_T \equiv \dfrac{v_T^{\mathrm{I}}}{\bar{v}_T^{\mathrm{S}}} = \hat{\rho}$ as the payoff to replicate.

Fair value

- After calculations we find the **fair value of the correlation proxy** $\hat{\rho}$:

$$c_0 = E(c_T) = \boxed{\frac{v_0^{\mathrm{I}}}{\overline{v}_0^{\mathrm{I}}}}\ \boxed{\exp\left[\frac{4}{3}\left(\overline{\omega}_{\mathrm{S}}^2 - \overline{\omega}_{\mathrm{S}}\omega_{\mathrm{I}}\chi\right)T\right]}$$

Adjustment Factor

Implied Correlation

$\hat{\rho}_0^*$

- The **implied-to-fair correlation adjustment factor** is given as:

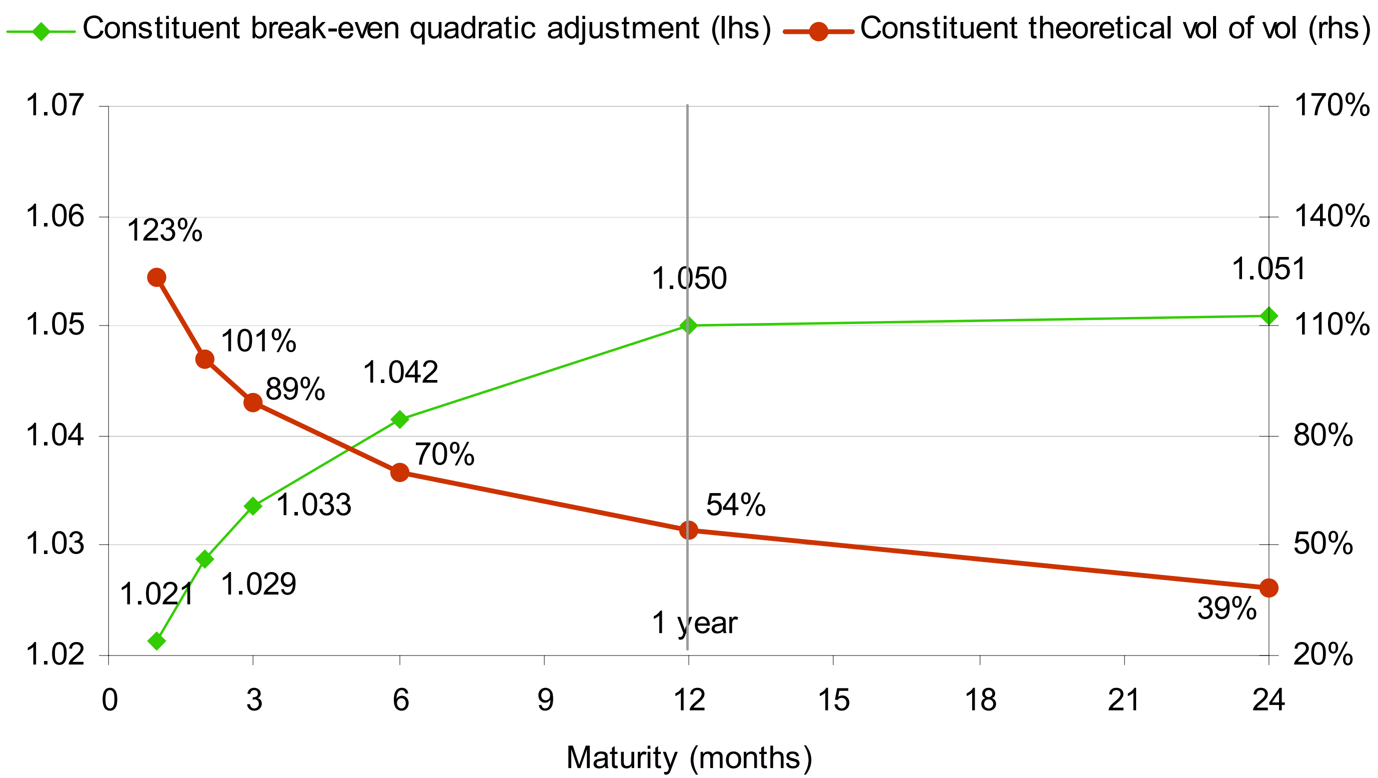
$$\frac{\hat{\rho}_0^*}{c_0} = \exp\left[\frac{4}{3}\left(\overline{\omega}_{\mathrm{S}}\omega_{\mathrm{I}}\chi - \overline{\omega}_{\mathrm{S}}^2\right)T\right]$$

- Note: For the adjustment factor to be above 1 (i.e. correlation swap strike below implied correlation, as observed on OTC markets), the correlation between index and constituent volatilities must be >> 0

3. Rational Pricing of Correlation Swaps

Parameter Estimation: Break-Even Analysis

- Break-even estimation of the volatility of constituent volatility of the DJ EuroStoxx 50 (2000—2005):



Implied-to-fair correlation adjustment factor: numerical examples

- Adjustment factor for various correlation of volatilities χ:

Mat.	Index volatility of volatility ω_I	Constituent volatility of volatility $\overline{\omega}_S$	Adjust. Factor (χ = 0.6)	Adjust. Factor (χ = 0.7)	Adjust. Factor (χ = 0.8)	Adjust. Factor (χ = 0.9)	Adjust. Factor (χ = 1)
1m	144.7%	123.4%	0.951	0.970	0.990	1.009	1.030
2m	122.6%	101.2%	0.940	0.966	0.993	1.021	1.049
3m	109.2%	88.9%	0.933	0.964	0.995	1.028	1.062
6m	86.5%	69.9%	0.920	0.957	0.997	1.038	1.081
12m	60.5%	54.1%	0.880	0.919	0.960	1.003	1.047
24m	41.5%	38.6%	0.869	0.906	0.946	0.987	1.031

3. Rational Pricing of Correlation Swaps

Dynamic Hedging Strategy

- Hedging coefficients (deltas):

$$\Delta_t^{\mathrm{I}} = \frac{c_t}{v_t^{\mathrm{I}}} \qquad \Delta_t^{\mathrm{S}} = -\frac{c_t}{\overline{v}_t^{\mathrm{S}}}$$

- Hedging portfolio:

Long index variance

Short constituent variance

$$\Pi_t = \Delta_t^{\mathrm{I}}.v_t^{\mathrm{I}} + \Delta_t^{\mathrm{S}}.\overline{v}_t^{\mathrm{S}} = \boxed{\boxed{\frac{c_t}{v_t^{\mathrm{I}}} v_t^{\mathrm{I}}} \boxed{-\frac{c_t}{\overline{v}_t^{\mathrm{S}}} \overline{v}_t^{\mathrm{S}}}} = \boxed{0}$$

Zero cost

Short vega-neutral variance dispersion
[Weight ratio between the constituent and index legs is equal to 'correlation']

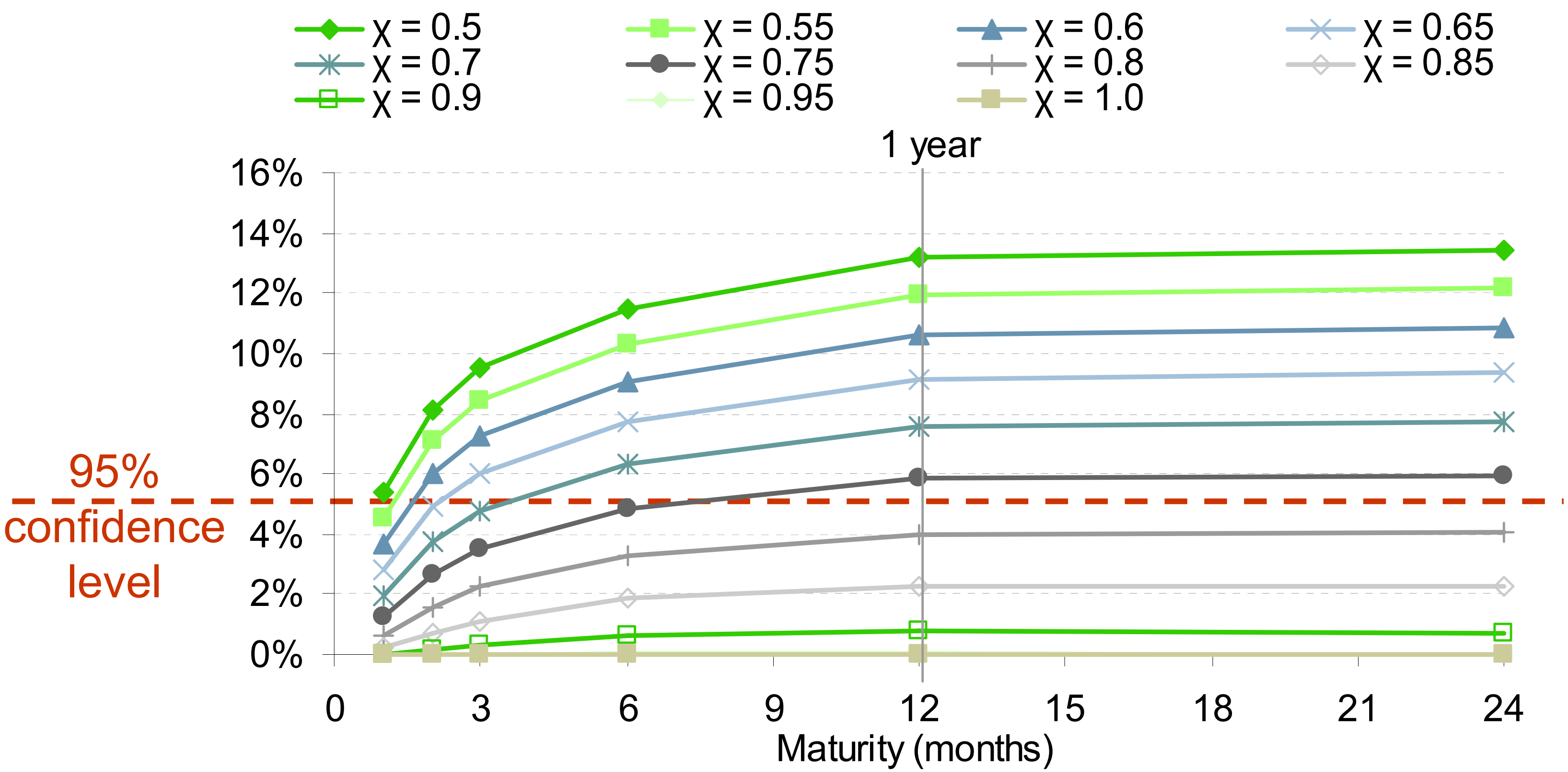
3. Rational Pricing of Correlation Swaps

Model limitations

- In addition to the limitations of the one-factor Toy Model, the **two-factor Toy Model is not entirely arbitrage-free** as a result of the unconstrained evolution of index and constituent variance price processes:
  - **The two-factor Toy Model allows for $v_t^I > v_t^S$!**
- Also the two-factor Toy Model relies on the assumption that **constituent stocks and their weights are static**, which is only reasonable for short maturities.

Model limitations

- Model probability of terminal realised correlation $c_T > 1$, for an initial implied correlation of 50%, *ad hoc* implied volatility of volatility parameters ω, and various correlation of volatilities χ:



Conclusion

- A correlation swap on an equity index can be **quasi-replicated** by **dynamically trading vega-neutral variance dispersions** at zero cost
- Using a straightforward extension of Black-Scholes, we find that **the fair strike of a correlation swap is equal to Implied Correlation multiplied by an adjustment factor** which depends on volatility of index volatility, volatility of constituent volatility and correlation between index and constituent volatilities.
- Using a parameter estimation methodology which relies on few historical observables, we obtain numerical results supporting the intuitive idea that **the adjustment factor should be close to 1**.

Further research

- **Fundamental**
  Toy Model needs to be made entirely arbitrage-free.
- **Practical**
  - Fair value of other correlation measures (e.g. canonical or average pair-wise measures)
  - Free-float weights, changes in index composition
- **Numerical**
  More sophisticated parameter estimations, over longer historical periods and in other markets

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References & Bibliography

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